

Black holes: interfacing the classical and the quantum¹

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Abstract

The central idea advocated in this paper is that forming the black hole horizon is attended with transition from the classical regime of evolution to the quantum one. We justify the following criterion for discriminating between the classical and the quantum: spontaneous creations and annihilations of particle-antiparticle pairs are impossible in the classical world but possible in the quantum world. We show that it is sufficient to change the overall sign of the spacetime signature in the classical picture of field propagation for it to be treated as its associated quantum picture. To describe a self-gravitating object at the last stage of its classical evolution, we propose to use the Foldy–Wouthuysen representation of the Dirac equation in curved spacetimes, and the Gozzi classical path integral. In both approaches, maintaining the dynamics in the classical regime is controlled by supersymmetry.

1 Introduction

The existence of black holes is one of the most intriguing predictions of general relativity (for a review see, e.g., [1, 2]). A black hole is defined as a region of spacetime from behind which it is impossible to escape to the future null infinity \mathcal{I}^+ without exceeding the speed of light. This region is invisible to distant observers. There is a boundary, called the event horizon, between an exterior region, where it is possible for signals to reach infinity, and the interior region, where signals remain trapped. Astrophysical black holes are assumed to arise as the final states of stellar evolution of sufficiently massive stars.

The formation of a black hole resembles a *phase transition* in condensed matter physics. The approach to black-hole states occurs fairly quickly. For a Schwarzschild black hole, the time scale for this approach is estimated at

$$\tau = \frac{c}{\kappa} = \frac{2r_s}{c}, \quad (1)$$

where κ is the surface gravity $\kappa = c^4/(4GM)$, and $r_s = 2GM/c^2$ is the Schwarzschild radius. Numerically, $\tau \simeq 2 \cdot 10^{-5}(M/M_\odot)$ s, where M_\odot is the mass of the Sun. If $M \sim M_\odot$, then $\tau \sim 10^{-5}$ s.

The *energy content* of the system in the initial state differs significantly from that in the final state. To see this, we refer to the Penrose process for extracting energy from a rotating and/or charged black hole. For the Kerr–Newman metric,

$$M^2 = \left(M_{\text{irr}} + \frac{Q^2}{4GM_{\text{irr}}} \right)^2 + \frac{J^2}{4G^2 M_{\text{irr}}^2}, \quad (2)$$

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where M is the total mass of the black hole, and M_{irr} is its irreversible mass. The maximum amount of rotational energy which can be extracted from an extreme Kerr black hole prior to slowing down its rotation is approximately 29% of its total energy. Furthermore, about 50% can be stored in the form of electromagnetic energy. By contrast, only few percent of nuclear binding energy can be emitted by stars throughout their lifetimes.

This transition entails a profound *symmetry rearrangement*. By the ‘no hair theorems’ [3, 4], each isolated, stationary black hole is completely described by three parameters: its mass M , angular momentum J , and electric charge Q . Whatever the structure of a star which collapses under its own gravitational field, the exterior of the resulting black hole is described by a Kerr–Newman solution. In other words, all initial symmetries of the collapsing system and their associated conservation laws, except for M , J , and Q , disappear in the ultimate black-hole state.

An appreciable distinction between this phenomenon and ordinary phase transitions lies in its *irreversibility*. Once converted into a black hole, the system can never regain its previous state.

What is the nature of this phase transition? If we take as our basic paradigm a collapsing star which settles down to a stationary black hole, then a plausible assumption is that **forming the black hole horizon is a transition between the classical and quantum regimes of evolution**. Indeed, the initial incarnation of this system is definitely classical. On the other hand, the resulting black hole is a quantum object. Any black hole evaporates due to Hawking radiation. [5, 6]. We now refer to the spontaneous creation of particle-antiparticle pairs in a strong gravitational field near the black hole horizon. One member of a virtual pair can fall into the black hole while its partner escapes to infinity. We recognize Hawking radiation in these processes of pair creation and the subsequent particle escaping.

According to the Hawking–Penrose singularity theorems [7], the collapse terminates in a singularity (big crunch) where the curvature is infinite and the classical concepts of space and time lose their meaning. However, quantum-gravitational effects are expected to be dominant as one approaches the Planck scale $\ell_P = \sqrt{G\hbar/c^3} \simeq 1.6 \cdot 10^{-33}$ cm. Taking into account the impact of trans-Planckian modes radiated outward from the center, one can infer that the interior of the black hole is governed by the laws of quantum physics.

Therefore, a collapsing system runs through two phases. The initial phase, before the black hole has formed, may be defined as classical, while the final phase, after the hole has formed, may be defined as quantum.

This paper is organized as follows. In Sec. 2 we obtain two criteria for discriminating between the classical and the quantum. We then argue that the black hole horizon is just the geometrical layout required to interface the classical and quantum realms. To take a closer look at this demarcation, we should have a general strategy together with suitable techniques. In Sec. 3 we briefly review the Foldy–Wouthuysen approach to the description of a Dirac particle, which provides a convenient representation of the classical regime of evolution. We will see that the Foldy–Wouthuysen transformation is possible in some curved spacetimes with stationary geometries, specifically in the Schwarzschild manifold. Furthermore, the feasibility of an exact Foldy–Wouthuysen transformation for a given system turns out to be related to the existence of the supersymmetry properties of this system. In Sec. 4 we outline the classical path integral concept. We demonstrate that the classical path integral can be obtained from the Feynman path integral if we change time t for the ‘supertime’ $(t, \theta, \bar{\theta})$, and the phase space coordinates q and p for the super-phase space coordinates Q and P . This structure of the classical path integral

makes it clear that the classical–quantum phase transition is attended with supersymmetry violation. Section 5 summarizes our discussion.

2 The classical and the quantum

What is the difference between the classical and quantum views of the same dynamical entity? We address this question by comparing the properties of *particles* and *fields* in the classical and quantum descriptions. A convenient framework bringing together the classical and quantum treatments is provided by the path integral approach.

Let us begin with particles. A quantum-mechanical particle can be described by the Feynman path integral

$$K(\mathbf{z}_f, T | \mathbf{z}_i, 0) = \int [\mathcal{D}q] \exp \left[\frac{i}{\hbar} \int_0^T dt L(q, q') \right]. \quad (3)$$

Here, $[\mathcal{D}q]$ means integration is to be carried out in the space of all paths from $\mathbf{z}_i(0)$ to $\mathbf{z}_f(T)$.

Whatever the kind of the world line $z^\mu(\tau)$ passing through the points $x^\mu = (0, \mathbf{z}_i)$ and $x^\mu = (T, \mathbf{z}_f)$ (as in Figure 1), it makes a contribution to the Feynman path integral provided that the Lagrangian L is real, and expression (3) is well defined for this path.

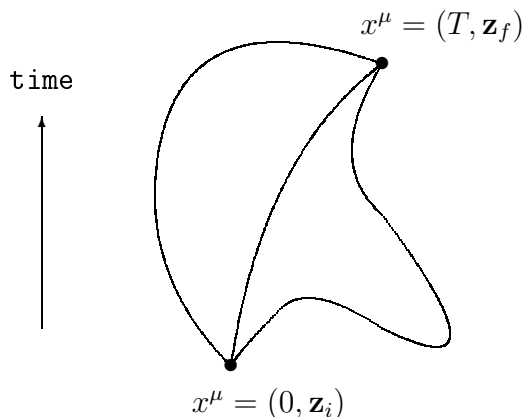


Figure 1: World lines contributing to the Feynman path integral

To illustrate, the Poincaré–Planck Lagrangian

$$L = -m \sqrt{\dot{z}^2} \quad (4)$$

is real and finite only for timelike paths. If $z^\mu(\tau)$ is a null curve, then $L = 0$. If $z^\mu(\tau)$ is spacelike, then L is complex-valued. Since the imaginary part of L can take both positive and negative values, expression (3) is ill-defined. By contrast, the Lagrangian proposed by Brink–Deser–Zumino–Di Vecchia–Howe [8]

$$L = -\frac{1}{2} \left(\eta \dot{z}^2 + \frac{m^2}{\eta} \right) \quad (5)$$

is real and finite for timelike, null, and spacelike curves. Here η is an auxiliary dynamical variable, sometimes called einbein.

We now look at Λ - and V -shaped world lines. Let a particle be moving along a timelike world line from the remote past to the future up to the point A, and then returns to the remote past. This Λ -shaped world line of a single particle can be interpreted as that representing the annihilation of a pair that occurs at a point A, because the antiparticle of this particle may be thought of as an object identical to it in every respect but moving back in time. Likewise, given a V -shaped world line of a single particle that runs initially

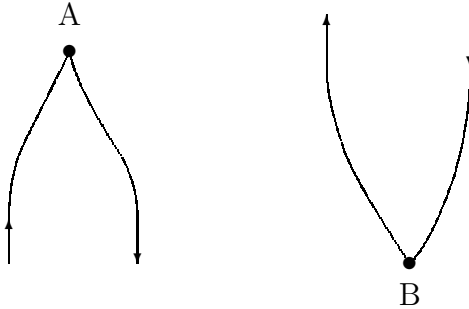


Figure 2: Λ - and V -shaped paths

from the far future to the past up to the point B and then returns to the far future, we interpret it as that representing the birth of a pair occurring at a point B. Any Λ - or V -shaped world line passing through the chosen end points $(0, \mathbf{z}_i)$ and (T, \mathbf{z}_f) contributes to the Feynman path integral (3). The quantum description leaves room for both particles and antiparticles together with their creations and annihilations.

On the other hand, classical particles are governed by the principle of least action. This principle can be formulated for smooth timelike and null world lines. However, it defies unambiguous formulation for V - and Λ -shaped world lines. Indeed, consider a spacelike hyperplane which represents space at some instant in a particular Lorentz frame. This hyperplane intersects a Λ -shaped curve twice, otherwise it fails to intersect this curve at all. The same is true for V -shaped curves. Thus, although the classical picture allows the coexistence of particles and antiparticles, creations and annihilations of pairs, represented by V - and Λ -shaped world lines, are banned [9].

Therein lies the fundamental difference between the quantum and classical viewpoints on particles: **spontaneous creations and annihilations of pairs are permissible in the quantum reality and impermissible in the classical reality**².

Hawking radiation, associated with the pair creation in a strong gravitational field near the black hole horizon, is a characteristically quantum phenomenon.

We now turn to fields. Having in mind the path integral approach, the key difference between the classical and quantum manifestations of the same field is due to the different boundary conditions imposed on their propagation laws. To be specific, consider a massless scalar field. The Fourier transform of the retarded Green's function

$$\tilde{D}_{\text{ret}}(k) = -\frac{1}{k^2 + 2ik_0\epsilon} = \frac{1}{\mathbf{k}^2 - (k_0 + i\epsilon)^2} \quad (6)$$

²It may be worth pointing out that behind this criterion is the fact that quantum theory gets by with a well defined action (appearing in the Feynman path integral) while classical theory requires additionally that the action be suited to its extremization following Hamilton's principle.

gives an accurate account of how this field propagates in classical theory. If the integration over the variable $\varkappa = |\mathbf{k}|$ is carried out first³, then the poles at

$$\varkappa = \pm\sqrt{k_0^2} \pm i\epsilon \quad (7)$$

are avoided by the path depicted in the left plot of Figure 3. Indeed, the pole with the positive real part lies above the integration path and the pole with the negative real part lies under the path. If the poles approach the real axis, the integration path should be slightly deformed to form the curve \mathcal{C}_{ret} in the complex \varkappa -plane.

The propagation of a free masses field in quantum theory is described by the Feynman propagator

$$\tilde{D}_F(k) = -\frac{1}{k^2 + i\epsilon}, \quad (8)$$

which obeys the causal boundary condition. It follows the prescription for avoiding the poles

$$\omega = \pm\sqrt{\mathbf{k}^2} \mp i\epsilon, \quad (9)$$

where ω denotes k_0 . The integration contour \mathcal{C}_F in the complex ω -plane is depicted in the right plot of Figure 3. Exact propagators of interacting quantum fields are given by the spectral Källén–Lehmann representation obeying the same boundary condition.

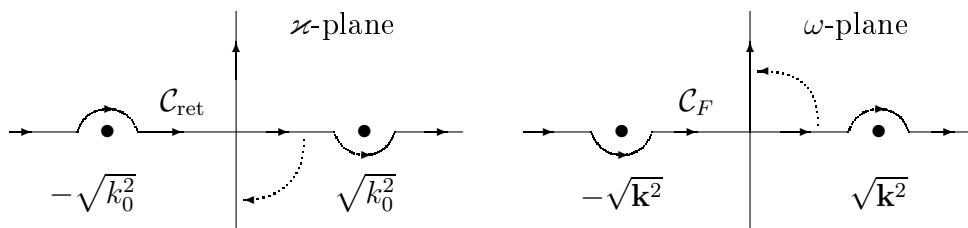


Figure 3: The integration contours \mathcal{C}_{ret} and \mathcal{C}_F suitable for \tilde{D}_{ret} and \tilde{D}_F

In order to bridge the gap between the retarded and causal boundary conditions, we ‘euclideanize’ both descriptions.

Assuming that the integrand decreases sufficiently fast as \varkappa approaches infinity, it is possible to rotate the path of integration \mathcal{C}_{ret} in a clockwise direction by $\frac{\pi}{2}$ in the complex \varkappa -plane without crossing the poles, as in left plot of Figure 3. This operation, the analytical continuation to imaginary values of the complex \varkappa -plane, is similar to the Wick rotation. Introducing a new variable $\mathbb{K} = i\mathbf{k}$ makes the length squared of k^μ positive definite:

$$k_E^2 = k_0^2 + \mathbb{K}^2. \quad (10)$$

The analytical continuation of the space variables to the imaginary axes

$$\mathbb{X} = i\mathbf{x}, \quad (11)$$

performed together with the analytical continuation in \mathbf{k} -space, introduces the Euclidean metric

$$dx_E^2 = dx_0^2 + d\mathbb{X}^2. \quad (12)$$

³We suppose that angular variables, if any, have already been integrated out, and that the resulting expression can be uniquely continued to the negative \varkappa -semiaxis.

On the other hand, if we carry out the Wick rotation of the path of integration \mathcal{C}_F in a counterclockwise direction by $\frac{\pi}{2}$ in the complex ω -plane without crossing the poles, as in right plot of Figure 3, which is equivalent to introducing $k_4 = ik_0$, then the length squared of k^μ becomes negative definite:

$$k_E^2 = -(k_4^2 + \mathbf{k}^2). \quad (13)$$

The analytical continuation of the time variable to the imaginary axis

$$x_4 = -ix_0, \quad (14)$$

performed together with the Wick rotation, introduces the Euclidean metric

$$dx_E^2 = -(dx_4^2 + d\mathbf{x}^2). \quad (15)$$

We thus see that it is sufficient to **change the overall sign of the spacetime signature in the classical description of field propagation for it to be treated as the quantum description of field propagation**. Indeed, two Lorentzian metrics with opposite signatures can always be analytically continued to two Euclidean line elements of opposite sign, such as those shown in (12) and (15).

Taken alone, the overall sign of the Lorentzian metric is of no particular importance, its choice is a matter of convention. In fact, the HEP theorists prefer the mainly negative signature $(+ - - -)$, while the relativists like the mainly positive signature $(- + + +)$. However, if this overall sign is changed as one passes from some region of spacetime to a contiguous region, then this change of sign is evidence of switching between the classical and quantum regimes of field propagation.

Such is the case for the contiguous regions inside and outside the event horizon of a black hole. As the simplest example, we refer to the Schwarzschild metric describing an isolated spherically symmetric stationary black hole

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 d\Omega. \quad (16)$$

Here, $d\Omega$ is the round metric in S^2 , and $r_S = 2GM/c^2$ is the Schwarzschild radius which represents the event horizon of this black hole. In the Schwarzschild exterior $r > r_S$, the Killing vector field $X = \partial_t$ is interpreted as the asymptotic time translation. In the Schwarzschild interior $r < r_S$, r is a time coordinate, and the integral lines of the vector field $X = \partial_r$ are incomplete timelike geodesics which terminate at $r = 0$. Once the Euclideanization has been performed, the regions inside and outside the boundary $r = r_S$ take the Euclidean metrics of the type of (12) and (15), respectively. The question now arises of what happens to the physical reality at the surface of the collapsing star when the Schwarzschild radius $r = r_S$ is crossed. Does the classical picture give way to the quantum picture, say, all cats which populate this star become Schrödinger's cats?

It is well known that the surface $r = r_S$ is locally perfectly regular. The singularity at $r = r_S$ is a mere coordinate singularity in the original Schwarzschild coordinate frame. In some other coordinates (such as Kruskal–Szekeres coordinates), the metric is smooth at $r = r_S$. When crossing $r = r_S$, an observer on the surface of a collapsing star will feel no geometrical and dynamical jumps. However, globally, $r = r_S$ acts as a point of no return. Furthermore, every light cone tilts over at this point, so that the roles of t and r are interchanged—no matter what the coordinate frame is used. Therefore, the

phase transition related to forming the Schwarzschild black hole should be viewed as a global concept: the entire spacetime must be known before its existence and form can be determined.

Consider a rotating black hole. The Kerr metric in Boyer–Lindquist coordinates reads

$$ds^2 = \frac{\Delta - a^2 \sin^2 \vartheta}{\Sigma} dt^2 + \frac{2a \sin^2 \vartheta (r^2 + a^2 - \Delta)}{\Sigma} dt d\varphi - \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta}{\Sigma} \sin^2 \vartheta d\varphi^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\vartheta^2, \quad (17)$$

where $\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta = r^2 + a^2 - 2GMr/c^2$. If $0 \leq a < GM/c^2$, then there are two distinct event horizons at $r = r_{\pm}$ where r_{\pm} are the roots of $\Delta = 0$,

$$\Delta = r^2 + a^2 - \frac{2GMr}{c^2} = (1 - r_-)(1 - r_+) = 0. \quad (18)$$

If $a = GM/c^2$, then there is a single event horizon. If $a > GM/c^2$, then the event horizon is nonexistent, and the curvature singularity is bare.

Thus, the analysis becomes more involved when the black hole rotation is incorporated. Things get worse in particular because there is no known explicit solution with rotating matter whose final state is described by a stationary geometry with the Kerr metric.

Let us now pass over charged black holes described by the Reissner–Nordstrom and Kerr–Newman solutions. Despite further technical subtleties, the major conclusion that such objects are suitable to studying a classical-quantum phase transition still stands.

We close this section by noting that a black hole horizon shows a clear demarcation boundary between spacetime regions characterized by opposite signatures. It seems likely that **this geometrical layout, if it exists, is the only opportunity for interfacing the classical and the quantum.**

3 The Foldy–Wouthuysen picture

A convenient framework for analyzing the classical and quantum regimes of evolution from a unified perspective is provided by the Dirac equation. Foldy and Wouthuysen [10] found a unitary transformation which diagonalizes the free Dirac Hamiltonian H_0 with respect to positive and negative energies,

$$U_{\text{FW}}^{-1} H_0 U_{\text{FW}} = \begin{pmatrix} \sqrt{-\nabla^2 + m^2} & 0 \\ 0 & -\sqrt{-\nabla^2 + m^2} \end{pmatrix} = \beta |H_0|. \quad (19)$$

From here on we will use the natural system of units in which $c = 1$ and $\hbar = 1$. The standard representation of Dirac matrices is assumed,

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (20)$$

where I is the 2×2 unit matrix, and σ^i are Pauli matrices.

Following the conventional interpretation, we consider positive energy states as those attributed to a Dirac particle, while states of negative energy are those attributed to its antiparticle. Thus, the free Dirac equation

$$i \frac{\partial}{\partial t} \psi = [i (\vec{\alpha} \cdot \nabla) + \beta m] \psi \quad (21)$$

is unitarily equivalent to a pair of two-component equations

$$i \frac{\partial}{\partial t} \chi = \beta \sqrt{-\nabla^2 + m^2} \chi. \quad (22)$$

An important point is that a separation of positive- and negative-energy states is still possible in the presence of an external time-independent magnetic field of arbitrary strength \mathbf{B} . Case [11] obtained a closed form for the Foldy–Wouthuysen transformation of the Dirac equation into

$$i \frac{\partial}{\partial t} \chi = \beta \sqrt{(i\nabla - e\mathbf{A})^2 - e(\vec{\sigma} \cdot \mathbf{B}) + m^2} \chi, \quad (23)$$

where \mathbf{A} is the vector potential of this field, $\mathbf{B} = \text{curl } \mathbf{A}$. The energy gap of the Dirac sea is not penetrated in the constant magnetic field because this field leaves the energy of the Dirac particle unchanged.

However, the separation is not possible with time-dependent electromagnetic fields and scalar potentials; the positive- and negative-energy solutions may mix when the interaction is sufficiently strong (a general review of these and related problems can be found in [12]–[14]). The energy gap of the Dirac sea is no longer insuperable. Creations and annihilations are made possible.

We interpret these facts by saying that a Dirac particle manifests itself as a classical entity in the case that the Hamiltonian can be diagonalized with respect to positive and negative energies.

Is it possible to transform the Dirac equation in a curved spacetime to the Foldy–Wouthuysen form? Obukhov [15] showed that an exact Foldy–Wouthuysen transformation U_{FW} can be written for stationary metrics

$$ds^2 = V^2(\mathbf{r}) dt^2 - W^2(\mathbf{r}) d\mathbf{r}^2, \quad (24)$$

where V and W are arbitrary functions of spatial coordinates $\mathbf{r} = (x, y, z)$. Schwarzschild geometry is a particular case of (24). When employing isotropic coordinates, the metric (16) takes the form (24) with

$$V = \left(1 - \frac{r_S}{4r}\right) \left(1 + \frac{r_S}{4r}\right)^{-1}, \quad W = \left(1 + \frac{r_S}{4r}\right)^2. \quad (25)$$

To be more specific, we recall that Dirac particles in curved backgrounds are governed by the covariant Dirac equation

$$(i\gamma^{\hat{\alpha}} D_{\hat{\alpha}} - m) \psi = 0. \quad (26)$$

Here, $D_{\hat{\alpha}}$ is the spinor covariant derivative

$$D_{\hat{\alpha}} = e^{\mu}_{\hat{\alpha}} \left(D_{\mu} + \frac{1}{4} \Gamma_{\mu} \right), \quad (27)$$

with the gravitational gauge potential $\Gamma_{\mu} = \frac{1}{4} [\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}] \gamma_{\hat{\alpha}\hat{\beta}\hat{\gamma}} e^{\hat{\gamma}}_{\mu}$ and the Ricci rotation coefficients $\gamma_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = e^{\mu}_{\hat{\alpha}} e_{\hat{\beta}\mu;\nu} e^{\nu}_{\hat{\gamma}}$. If (26) can be brought into the form

$$i \frac{\partial \psi}{\partial t} = H \psi \quad (28)$$

with

$$H = \beta m - \frac{i}{2} [(\vec{\alpha} \cdot \nabla)F + F(\vec{\alpha} \cdot \nabla)], \quad F = \frac{V}{W}, \quad (29)$$

then there is a unitary transformation $H_{\text{FW}} = U_{\text{FW}} H U_{\text{FW}}^\dagger$ such that

$$H_{\text{FW}} = \frac{1}{2} \left(\sqrt{\bar{H}^2} + \beta \sqrt{\bar{H}^2} \beta \right) + \frac{1}{2} \left(\sqrt{\bar{H}^2} - \beta \sqrt{\bar{H}^2} \beta \right) \gamma_5 \beta, \quad (30)$$

$$\bar{H}^2 = m^2 V^2 - F \nabla^2 F + \frac{1}{2} F (\nabla \vec{f}) - \frac{1}{4} \vec{f}^2 - i F \vec{\Sigma} \cdot [\vec{f} \times \nabla - \gamma_5 \beta m \vec{\phi}]. \quad (31)$$

Here, $\vec{\Sigma}$ is the spin matrix $\vec{\Sigma} = \frac{1}{2} i \vec{\gamma} \times \vec{\gamma}$, and $\vec{f} = \nabla F$, $\vec{\phi} = \nabla V$.

A remarkable fact is that the feasibility of an exact Foldy–Wouthuysen transformation is another way of stating that the system enjoys the property of supersymmetry [16]. The origin of the Foldy–Wouthuysen picture for a Dirac particle in an external electromagnetic field is related to the existence of a supercharge. If the Dirac sea is stable, then the positive- and negative-energy solutions are supersymmetric partners of each other. On the other hand, when the supersymmetry is broken, it is impossible to obtain an exact block-diagonalized Hamiltonian for this system.

These findings can be extended to curved spacetimes [17]. A supercharge can be constructed for a relatively wide class of stationary metrics, including that defined in (24). The Foldy–Wouthuysen transformed Hamiltonian H_{FW} is proportional to the square root of the super-Hamiltonian

$$H_{\text{FW}} = \beta \sqrt{Q^2}. \quad (32)$$

For the metric (24),

$$Q = \frac{1}{2} \{ \vec{\alpha} \cdot p, F \} + JV. \quad (33)$$

Here, J is the involution operator $J = i\gamma_5 \beta$ (a Hermitian and unitary operator, $J^\dagger = J$, $JJ^\dagger = 1$, which anticommutes with both the Hamiltonian and the β matrix, $JH + HJ = 0$, $J\beta + \beta J = 0$).

We now come to the following problem. Consider a self-gravitating Dirac field ψ which arranges itself into a spherically symmetric collapsing wave packet. Let the total mass of the wave packet be equal to M , the parameter which enters in the definition of the Schwarzschild radius r_s . Before a black hole state settles down, the ψ is assumed to model an astrophysical collapsing object in the Schwarzschild spacetime. Our interest here is with the ‘last stage’ of evolution of the wave packet just before its shrinking down to below the horizon. At this stage, the ψ is governed by the diagonalized Foldy–Wouthuysen Hamiltonian (30). It would be desirable to have an exact solution to this problem. Perhaps this solution will exhibit a singular point after which this classical regime of evolution is no longer valid. Note that this singularity is just the point in which the supersymmetry must be violated.

Of course a similar discussion of rotating spacetimes, say, based on the Kerr geometry, would be highly desirable. This would provide insight into the mechanism of forming a black hole from a collapsing fluid ψ endowed with spin degrees of freedom.

4 The classical path integral

Not only quantum mechanics, but also classical mechanics can be treated through a path integral formulation proposed by Gozzi [18], and developed in a long series of papers by

Gozzi and his collaborators (see [19, 20] and references therein). We now outline the basic elements of this treatment.

The probability amplitude of finding a classical system at a phase space point $\phi_f^a = (q_f^a, p_f^a)$ at time $t_f = T$ if it was at $\phi_i^a = (q_i^a, p_i^a)$ at time $t_i = 0$ is given by

$$K(\phi_f^a, T | \phi_i^a, 0) = \int [\mathcal{D}\phi] \delta[\phi^a - \phi_{\text{cl}}^a(T; \phi_i, 0)]. \quad (34)$$

Here, ϕ_{cl}^a is the solution to the classical equation of motion $\dot{\phi}^a = \omega^{ab} \partial_b H$, with ω^{ab} being a symplectic matrix, and H the Hamiltonian of this system. The symbol $[\mathcal{D}\phi]$ indicates that the integration is over the all phase space paths with fixed end points ϕ_i and ϕ_f .

Since

$$\delta(\phi - \phi_{\text{cl}}) = \delta\left(\dot{\phi}^a - \omega^{ab} \partial_b H\right) \det(\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H), \quad (35)$$

one may further take the Fourier transform of the Dirac delta and exponentiate the determinant using an even Grassmannian variable λ_a and odd variables c^a and \bar{c}_a to yield

$$K(\phi_f, T | \phi_i, 0) = \int [\mathcal{D}\phi] \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \exp\left(i \int_0^T dt \tilde{\mathcal{L}}\right). \quad (36)$$

where

$$\tilde{\mathcal{L}} = \lambda_a \dot{\phi}^a + i \bar{c}_a \dot{c}^a - \lambda_a \omega^{ab} \partial_b H - i \bar{c}_a \omega^{ad} \partial_d \partial_b H c^b. \quad (37)$$

If we define two anticommuting partners of t , $\bar{\theta}$ and θ , and assemble the variables $\phi, \lambda, \bar{c}, c$ into a single combination of supersymmetric phase space coordinates

$$Q(t, \theta, \bar{\theta}) = q(t) + \theta c^q + \bar{\theta} \bar{c}_p + i \bar{\theta} \theta \lambda_p, \quad (38)$$

$$P(t, \theta, \bar{\theta}) = p(t) + \theta c^p - \bar{\theta} \bar{c}_q - i \bar{\theta} \theta \lambda_q, \quad (39)$$

then it is possible to rewrite (36) in a very compact and elegant form [20]:

$$K(Q_f, T | Q_i, 0) = \int [\mathcal{D}Q] \mathcal{D}P \exp\left[i \int_0^T dt d\theta d\bar{\theta} L(Q, P)\right], \quad (40)$$

where L is the usual Lagrangian of this system $L(q, p) = p\dot{q} - H(q, p)$. Equation (40) bears the formal similarity to the quantum path integral

$$K(q_f, T | q_i, 0) = \int [\mathcal{D}q] \mathcal{D}p \exp\left[\frac{i}{\hbar} \int_0^T dt L(q, p)\right]. \quad (41)$$

In fact, (40) derives from (41) by replacing the phase space coordinates q, p with the super phase space coordinates Q, P and extending the integration over t to an integration over the supertime $(t, \theta, \bar{\theta})$, multiplied by \hbar .

Turning back to our main subject, consider a gravitating perfect fluid in a Schwarzschild background, which arranges itself into a collapsing ball. The canonical theory of classical perfect fluids, in Eulerian and Lagrangian formulations, is well studied (a review can be found in [21]). It is then desirable to extend this theory to curved spacetimes, construct its supersymmetric version by substituting the phase space for the super phase space, as shown in (38)–(39), and write the classical path integral (40). If we would succeed in working out this integral, then we would read from the resulting expression a self-denial of the classical physics at some point. Notice that the supersymmetry structure of equation (40) is automatically broken at this point.

5 Conclusion

Let us summarize the results of our discussion.

The black hole formation bears some resemblance to a phase transition. The approach to the black-hole state occurs fairly quickly. The energy content of the system in an initial state differs significantly from that after the hole has formed. This transition entails a profound symmetry rearrangement: by the no hair theorems, all initial symmetries and their associated conserved quantities, except for M , J , and Q , disappear in the ultimate black-hole state.

The central idea of this paper is that forming the black hole horizon implies a transition between the classical and quantum regimes of evolution.

When the essentials of classical theory is compared with those of quantum theory, it transpires that spontaneous creations and annihilations of particle-antiparticle pairs are impossible in the classical world, but possible in the quantum world. Furthermore, it is sufficient to change the overall sign of the spacetime signature in the classical description of field propagation for it to be treated as the quantum description of field propagation. This leads us to consider the black hole horizon as a sharply defined boundary which demarcates the classical and the quantum.

The feasibility of the classical regime of evolution is another way of stating that the system enjoys the property of supersymmetry. To describe a self-gravitating object at the last stage of its classical evolution, just before its shrinking down to below the horizon, we proposed to invoke the Foldy–Wouthuysen representation of the Dirac equation in curved spacetimes, and/or the Gozzi classical path integral technique. In both descriptions, maintaining the dynamics in the classical regime is controlled by supersymmetry. If we would succeed in integrating the Foldy–Wouthuysen dynamics for a collapsing wave packet ψ (say, in the Schwarzschild background), or, alternatively, in calculating the Gozzi path integral for a gravitationally collapsing perfect fluid, we would establish a point where a self-destruction of this classical machinery occurs. The supersymmetry structures undergoes a breakdown at this point.

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